Int. J. Solids Structures Vol. 15, pp. 861–870
Pergamon Press Ltd., 1979. Printed in Great Britain

KINEMATIC HARDENING RULE IN SINGLE CRYSTALS

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(*Received* 5 *October* 1978; *in revised form* 26 *January* 1979; *received for publication* 2 *April* 1979)

Abstract—Based on the theories of Seeger's dislocation pile-up and Orowan's dispersion hardening, the hardening behavior of single crystals was found to obey Prager's kinematic hardening rule. Under single slip, yield surface moves along its normal while under multislip it moves along the direction of incremental stress. The theory developed is consistent with the Bauschinger effect; it is also in good agreement with the observations of Edwards and Washburn on the latent hardening of zinc, cadmium and copper crystals.

1. INTRODUCTION

The derivation of the elastic–plastic behavior of a polycrystalline aggregate from those of its constituent single crystals is a fundamental problem in metal plasticity. An accurate derivation requires the utilization of both correct single crystal properties and a physically consistent mathematical method which can properly account for the grain interactions. Though several methods have been proposed in the past[1–7], relatively few theories on the hardening law of single crystals have been introduced. The existing hardening laws include Taylor's isotropic hardening[1], Koiter's "independent" hardening[8] and Hill's general "interdependent" hardening[9]. The independent hardening law is physically inconsistent. The isotropic hardening law is attractive in its simplicity and certain physical justifications under proportional loading. But because it is in direct contradiction to the Bauschinger effect[10, 11], it can lead to serious errors under reversed and non-radial loading conditions. Recently, based on Orowan's classic dispersion hardening[12, 13] a new hardening law was proposed by Weng and Phillips[14]. This new law is consistent with the Bauschinger effect; it falls within the general framework of Hill's interdependent hardening law. This theory, however, considers only single slip. Since crystal grains in a polycrystalline aggregate easily undergo multiple slip it is important that the theory developed be extended into this condition. This is the prime objective of this paper. In addition to Orowan's mechanism, Seeger's dislocation pile-up[15] will also be incorporated in the new hardening law. Finally to place the present theory in proper perspective the developed hardening law will be compared with Prager's kinematic hardening rule[16]. Our consideration will be limited to small deformation so that crystal rotation can be neglected.

2. SEEGER'S DISLOCATION PILE-UP AND ITS APPLICATION TO WENG-PHILLIPS' HARDENING LAW

We reiterate that our objective here is to develop a hardening law for single crystal grains so that the properties of a polycrystalline aggregate can be better understood. Single crystals behave differently in an aggregate. In order to remain compatible with their neighboring grains the prolonged "easy glide" region is greatly shortened, and in most instance simply disappears. What is particularly relevant to the theory of work hardening is the presence of dislocation pile-ups against the grain boundaries, in addition to the pile-ups against the large, strong obstacles. The additional pile-up against the grain boundaries has added an important factor that Seeger's pile-up theory is especially applicable here. Orowan's dispersion hardening is important in the fact that engineering metals generally contain impurities or precipitates. It is believed that the combination of these two mechanisms would provide a more adequate account on the hardening behavior of crystal grains.

In their study of the origins of work hardening in both face-centered-cubic and hexagonal-close-packed crystals Seeger and his associates[15, 17] attributed strain hardening to two factors: the long range, temperature independent dislocation pile-up (τ_G) and the short range, temperature dependent dislocation intersections (τ_S). It was found that although τ_S and τ_G are of comparable magnitude in the early Stage I the relative contribution of τ_S to the flow stress is substantially reduced in the early Stage II, and becomes almost negligible at the end of this stage. Since crystal grains do not experience Stage I deformation dislocation pile-up is considered to be the main source of work hardening.

Consider a fully annealed crystal. When the resolved shear stress τ on a slip system is below the critical shear stress τ_0, extensive dislocation movement does not occur (dislocations can be generally assumed to remain stationary), and deformation is elastic (lattice deformation). As τ reaches τ_0 slip or dislocation movement begins to take place. The presence of grain boundaries and large obstacles prevent the continued dislocation motion and results in the dislocation pile ups (see Fig. 1). At low and intermediate temperature dislocation climb process is not significant enough to remove this pile-up configuration. Because of the dislocation pile-up a back stress is developed. For continued dislocation movement this back stress has to be overcome by the applied stress and this results in the work hardening. At some stage of plastic deformation the flow stress has increased from τ_0 to $\tau_0 + \Delta\tau$. The configuration of the pile-up is characterized by the average dislocation spacing; closer spacing requires higher $\Delta\tau$ value.

Now if the applied stress is gradually released unloading takes place. Unloading is not accompanied with a reversed dislocation motion until the yield condition in the reversed direction is satisfied. It is known that the concept of back stress was first introduced by Taylor[18]. Based on the equilibrium consideration Orowan[13] argued that according to Taylor's model reversed dislocation movement will take place as soon as the applied stress is released. This in principle will not occur if a threshold stress, such as τ_0, is required for any significant dislocation motion. In reality according to Seeger[15] some sessile dislocations (e.g. Lomer–Cottrell dislocations in f.c.c. crystals) can develop at the tail of the pile-up groups and stabilize the piled-up dislocations from immediate release upon unloading. In view of the fact that the resultant acting force on the piled-up dislocations is still along the forward direction upon immediate unloading, it is unlikely that any significant reversed dislocation motion will take place along the reversed direction. Much like the concept that the developed back stress tends to resist the dislocations with an effective hardening $\Delta\tau$, the repulsive force will tend to push the piled-up dislocations to move in the reversed direction[19]. With this back stress, the yield condition in the reversed direction is satisfied when $\tau = -(\tau_0 - \Delta\tau)$. The hardening behavior in the active slip system associated with Seeger's pile-up theory is seen to be "kinematic" in nature.

The kinematic nature of Orowan's dispersion hardening is attributed to the long-range back stress generated by the dislocation loops encircling the dispersions and the accumulation of weak obstacles in front of mobile dislocations. This property was discussed in [14].

The combination of Seeger's dislocation pile-up and Orowan's dispersion hardening leads to the condition that while dislocations move, on the one hand they push away the weak obstacles and pass by others leaving them encircled with dislocation loops; on the other they pile up against the grain boundaries and strong obstacles. Both mechanisms are kinematic in nature and result in the Bauschinger effect under reversed loading. These predictions are seen to be in good agreement, both quantitatively and qualitatively, with the observations of Edwards *et al.*[10] on cadmium crystal (see Fig. 2).

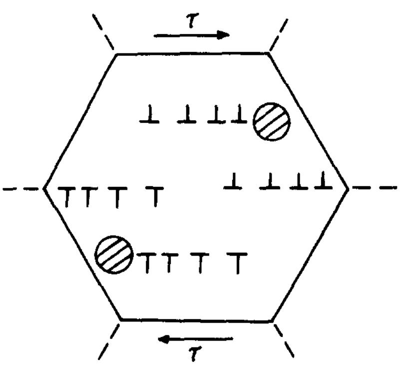

Fig. 1.

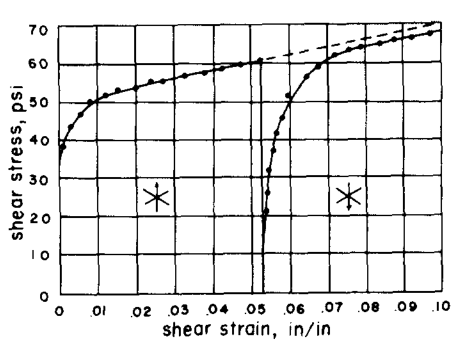

Fig. 2.

Fig. 1. Seeger's dislocation pile-up.

Fig. 2. Stress–strain curve of a cadmium crystal under reversed slip[10].

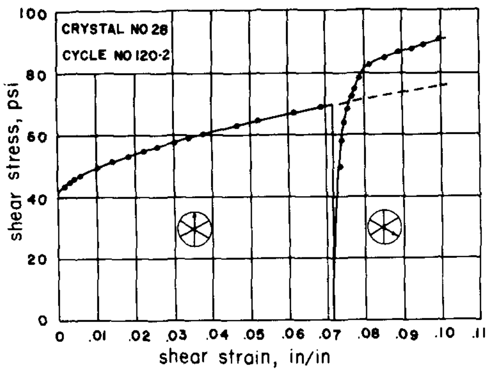

Fig. 3. Stress–strain curve of a zinc crystal under changing slip system[21].

Hardening not only occurs in the active slip system, it also occurs in the latent ones and this causes latent hardening. The latent hardening law associated with Seeger's theory can be established by considering the stress field of a dislocation pile-up and the Peach–Koehler force acting on a dislocation[20] due to this pile-up stress. An immediate consequence of this consideration is that, when the slip direction, or the slip plane normal of the latent system is perpendicular to that of the active one, the effective force vanishes. This consideration leads to an identical hardening law established in [14] based on Orowan's mechanism. This hardening law states that when the angle between the glide directions of the active and latent slip systems is denoted by θ, and that between their glide planes by ϕ, the amount of hardening in the latent slip system is given by†

$$\Delta\tau \cos\theta \cos\phi. \tag{1}$$

This hardening law is in good agreement with the observation of Edwards and Washburn[21] on the latent hardening of zinc crystal (see Fig. 3).

To facilitate our later formulation we recall some useful relations established in [14] here. When a crystal is under a combined stress σ_{ij}, the initial yield conditions of a slip system for the forward and reversed slips are respectively given by

$$Q_{2i}Q_{1j}\sigma_{ij} = \tau_0 \tag{2a}$$

$$Q_{2i}Q_{1j}\sigma_{ij} = -\tau_0 \tag{2b}$$

where $Q_{ij} \equiv \cos(x'_i, x_j)$, x'_1 and x'_2 being the slip direction and the slip plane normal, respectively, and x_i the material axes. At certain stage of plastic deformation the flow stress in the active slip system increases to $\tau_0 + \Delta\tau$ for the forward slip and decreases to $-(\tau_0 - \Delta\tau)$ for the reversed slip. Denoting the active slip system as the first system its yield conditions for these two slip directions are respectively given by

$$\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}\sigma_{ij} = \tau_0 + \Delta\tau \tag{3a}$$

$$\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}\sigma_{ij} = -(\tau_0 - \Delta\tau). \tag{3b}$$

The amount of active hardening $\Delta\tau$ is related to the incremental loading $\Delta\sigma_{ij}$ from the initial yield surface by

$$\Delta\tau = \overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}\Delta\sigma_{ij}. \tag{4}$$

†For the sake of simplicity and comparison with other hardening laws we shall neglect the effect of stress relaxation studied in [14] in this paper. Thus $\Delta\tau_c = \Delta\tau$.

The yield conditions for any latent slip system, generally called system 2, are

$$\overset{(2)}{Q}_{2i}\overset{(2)}{Q}_{1j}\sigma_{ij} = \tau_0 + \Delta\tau \cos \overset{(2,1)}{\psi} \tag{5a}$$

$$\overset{(2)}{Q}_{2i}\overset{(2)}{Q}_{1j}\sigma_{ij} = -(\tau_0 - \Delta\tau \cos \overset{(2,1)}{\psi}) \tag{5b}$$

where $\overset{(2,1)}{\psi}$ is the angle between the normals of the yield planes of the second and first slip systems, given by $\cos \overset{(2,1)}{\psi} = \cos\theta \cos\phi$.

In crystalline plasticity the combination of a slip plane and a slip direction on this plane is traditionally called a slip system. Slip can take place along the positive (forward) or the negative (reversed) direction as discussed above. This consideration provides one pair of yield planes to each slip system. Though this concept is useful to the understanding of the physics of plastic deformation, it is not convenient to the description of hardening rule, because the forward and reversed slips of the same slip system have to be treated separately.

To circumvent this inconvenience we redefine here that a slip system is consisted of a slip plane and *one* slip direction on this plane. The other (opposite) slip direction with the same slip plane is considered to form another slip system. With this new definition it can be shown that the subsequent yield planes of both active and latent slip systems, originally given by eqns (3a, b) and (5a, b) can be combined into

$$\overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}\sigma_{ij} = \tau_0 + \Delta\tau \cos \overset{(k,1)}{\psi} \tag{6}$$

where k refers to the kth slip system under the new definition. When the subsequent yield plane given by eqn (6) is plotted for all slip systems, the envelope of these planes gives rise to the subsequent yield surface of the single crystal. In the special case when $\Delta\tau = 0$ eqn (6) will give the initial yield surface. This definition will be adopted in the remaining sections of this paper.

3. SUBSEQUENT YIELD SURFACE UNDER MULTIPLE SLIP

Though some crystals (e.g. h.c.p.) can have a prolonged "easy glide" under some special loading condition, most others with higher slip systems (e.g. b.c.c. and f.c.c.) can readily undergo multiple slip. In particular for a crystallite to remain compatible with its neighboring grains five independent active slip systems are generally required. The hardening rule of a single crystal under multiple slip is particularly important to the study of the constitutive relations of a polycrystalline aggregate.

Extending the hardening law established in (1) into multislip condition, we find that when there are N simultaneous active slip systems, the latent hardening in a slip system is given by

$$\sum_{n=1}^{N} \tau_n \cos \overset{(n)}{\theta} \cos \overset{(n)}{\phi} \tag{7}$$

where τ_n is the active hardening of the nth active slip system, $\overset{(n)}{\theta}$ and $\overset{(n)}{\phi}$ refer to the same angles indicated above between the considered slip system and the nth active slip system.

For multiple slip to occur the yield conditions of at least two slip systems have to be satisfied simultaneously, and the applied stress has to lie at the corner of yield surface. Consider Fig. 4. When the single crystal is under an incremental loading $\Delta\sigma_{ij}$, denoted by vector $\overline{AB}$, the yield conditions of both first and second slip systems are satisfied. The questions are: where is the subsequent yield surface following such an incremental loading? Would eqn (6) still provide the subsequent yield surface? What are the values of $\Delta\tau_n$ in eqn (7)?

Under multiple slip (duplex slip in Fig. 4) the yield conditions of both active slip systems continue to be satisfied. The final stress state B thus again has to lie at the intersection of the subsequent yield planes of these two slip systems. Neglecting small grain rotation the sub-

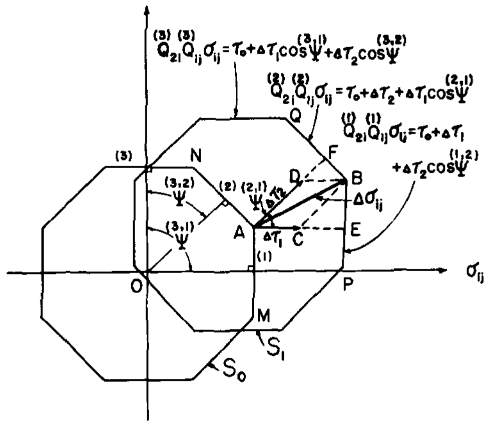

Fig. 4. Subsequent yield surface under multiple slip.

sequent yield planes PB and BQ can be obtained by parallel translation of the initial yield planes MA and AN to this position. It is evident from this figure that the total hardening in the first slip system is given by

$$AE = \overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}\Delta\sigma_{ij} \tag{8a}$$

and that in the second slip system it is given by

$$AF = \overset{(2)}{Q}_{2i}\overset{(2)}{Q}_{1j}\Delta\sigma_{ij}. \tag{8b}$$

Since both slip systems are active under $\Delta\sigma_{ij}$, each total hardening given in eqns (8) is consisted of two parts: the active hardening due to the dislocation movement of its own system and the latent hardening due to the dislocation movement of the other one. From the analysis of previous section the latent hardening of one slip system due to the active hardening of another system is given by $\Delta\tau \cos\psi$. If vector $\overline{AB}$ is decomposed into $\overline{AC}$ and $\overline{AD}$ along the normals of the yield planes, it is readily seen that

$$AE = AC + AD\cos\overset{(2,1)}{\psi} \tag{9a}$$

and

$$AF = AD + AC\cos\overset{(2,1)}{\psi}. \tag{9b}$$

The active hardening of slip system 1, denoted by $\Delta\tau_1$, is thus identified with AC, and that of slip system 2, denoted by $\Delta\tau_2$, is equal to AD. Equations (9) consequently become

$$\Delta\tau_1 + \Delta\tau_2\cos\overset{(1,2)}{\psi} = \overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}\Delta\sigma_{ij} \tag{10a}$$

$$\Delta\tau_2 + \Delta\tau_1\cos\overset{(2,1)}{\psi} = \overset{(2)}{Q}_{2i}\overset{(2)}{Q}_{1j}\Delta\sigma_{ij}. \tag{10b}$$

The values of active hardening $\Delta\tau_1$ and $\Delta\tau_2$ in these two active slip systems thus can be obtained from eqns (10) in terms of $\Delta\sigma_{ij}$ and $\overset{(2,1)}{\psi}$.

Since the active hardening of both slip systems contribute to the hardening of any latent system, denoted by the kth slip system, the latent hardening is given by

$$\Delta\tau_1 \cos \overset{(k,1)}{\psi} + \Delta\tau_2 \cos \overset{(k,2)}{\psi}. \tag{11}$$

The subsequent yield condition of the kth slip system consequently becomes

$$\overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}\sigma_{ij} = \tau_0 + \Delta\tau_1 \cos \overset{(k,1)}{\psi} + \Delta\tau_2 \cos \overset{(k,2)}{\psi}. \tag{12}$$

Equation (12) is the counterpart of eqn (6) under duplex slip. When k is considered for all slip systems the envelope of the planes is the subsequent yield surface of the single crystal, as denoted by S_1 in Fig. 4.

In general when the multiple slip involves N active slip systems (maximum 5), the total hardening in each of these systems is consisted of N parts: its own active hardening and the latent hardening due to the active hardening of the other N–1 slip systems. The values of these $\Delta\tau_n$, which satisfied eqn (10) under duplex slip, will satisfy the following N simultaneous equations

$$\sum_{n=1}^{N} \Delta\tau_n \cos \overset{(m,n)}{\psi} = \overset{(m)}{Q}_{2i}\overset{(m)}{Q}_{1j}\Delta\sigma_{ij} \tag{13}$$

where $m = 1, 2, \ldots N$. The values of $\Delta\tau_n$ can be readily solved in terms of $\Delta\sigma_{ij}$ and $\cos \overset{(m,n)}{\psi}$. The amount of hardening in the kth slip system attributed to the active hardening of these N slip systems is given by

$$\sum_{n=1}^{N} \Delta\tau_n \cos \overset{(k,n)}{\psi} \tag{14}$$

where k could be an active or a latent slip system. The subsequent yield plane of the kth slip system thus can be written as

$$\overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}\sigma_{ij} = \tau_0 + \sum_{n=1}^{N} \Delta\tau_n \cos \overset{(k,n)}{\psi}. \tag{15}$$

The envelope of the subsequent yield planes of all slip systems is the subsequent yield surface of the single crystal under N multiple slips.

Criterion of an "active" slip system at the corner:

When the incremental loading $\overline{AB}$ lies within the cone of the normals EAF as shown in Fig. 4, it can be decomposed into two vectors $\overline{AC}$ and $\overline{AD}$, each representing the active hardening of the associated slip system. Duplex slip thus can be visualized to take place as a sequence of two single slips, with slip system 1 active along $\overline{AC}$ and then system 2 along $\overline{CB}$, or vice versa. In each step the hardening rule of single crystal under single slip applies. Whether the path is taken to be along ACB or along ADB the same final subsequent yield surface is obtained. Under single slip since the incremental plastic strain is normal to its yield plane[14], the resultant plastic strain under duplex slip will lie within the cone EAF. Thus $d\sigma_{ij}\, d\epsilon_{ij}^{p} > 0$ for each single slip as well as for the duplex slip, and Drucker's stability postulate[22] is completely satisfied.

But when the incremental loading $\overline{AB}$ lies outside the cone EAF, the questions arise: will it cause duplex slip or single slip? If single slip which system prevails?

To answer these questions consider Fig. 5 in which vector $\overline{AB}$ lies outside the cone of the normals EAF and leans toward the first yield plane. Following the same process vector $\overline{AB}$ is decomposed into two components $\overline{AC}$ and $\overline{AD}$ along the normals of both yield planes. Notice

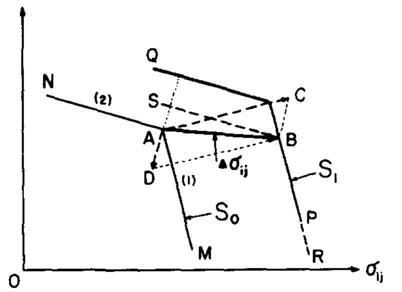

Fig. 5.

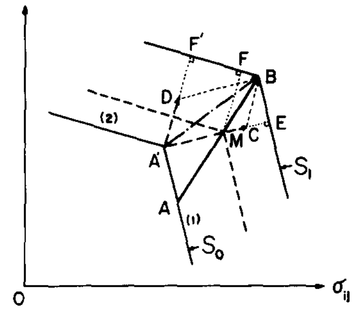

Fig. 6.

Fig. 5. Determination of "active" slip system at the corner.

Fig. 6. Change from single to multiple slip.

$\overline{AD}$ points inward. It is evident that though loading path $\overline{AC}$ can cause single slip for slip system 1 path $\overline{AD}$ is an unloading process for slip system 2. Should it cause the second system to slip it would have violated Drucker's postulate. This picture can also be examined in another way. Suppose that $\overline{AB}$ would cause duplex slip, then the subsequent yield planes of both active slip systems would have to pass through point B as shown by the dotted line RBS. But under such circumstance the total hardening on the second slip system would be less than its latent hardening due to slip system 1, and this is inconsistent to the nature of hardening since $\psi < \pi/2$. (In case $\psi > \pi/2$ the yield condition of slip system 2 at state B is obviously not satisfied.) It thus can be concluded that though at stress state A the yield conditions of both slip systems are satisfied, the incremtnal loading $\overline{AB}$ as shown in Fig. 5 causes only single slip in the first slip system.

In general when the yield condition of N slip systems are simultaneously satisfied at state A, to determine whether a slip system is actually active under an incremental loading $\overline{AB}$ acting at the corner of the yield surface, one can decompose vector $\overline{AB}$ into N components along the normals of these N yield planes. If this component is positive for some slip system, this system is active. Otherwise it is inactive. Mathematically this means that if

$$\Delta\tau_n > 0 \tag{16}$$

nth slip system is active, otherwise it is not. This concept is consistent with the "loading" criterion in continuum plasticity.

Transition from single to multiple slip:

Under radial (proportional) loading the initially active slip systems will remain active throughout the entire deformation. However under nonradial loading some active slip systems can become inactive and the inactive ones can become active. To construct the subsequent yield surface under such a condition, the incremental loading can be divided into several stages, each characterized by either a single slip or a multiple slip. The subsequent yield surface can be determined by treating each stage separately.

Consider an incremental loading $\overline{AB}$ which intersects the normal of the first yield plane $A'E$ at point M (see Fig. 6). From A to M system 1 is the only active slip system, whereas from M to B both systems 1 and 2 are active. The final subsequent yield surface can be obtained by applying the hardening rule of single slip during AM and the hardening rule of duplex slip during MB. If the orientation of $\overline{AB}$ is such that vector MB lies outside the cone EMF and leans toward the second yield plane, slip system 1 will cease to be active after M while system 2 becomes active. The hardening rule of single slip will have to be used for each stage of incremental loading.

In the event when the active slip system continues to be active while others are joing in, the above process can be further simplified. Since the overall deformation during $\overline{AB}$ involves the dislocation movement of two slip systems, this suggests that such a deformation might be treated as multiple slip. In view of the fact that the subsequent yield planes of all active slip systems pass through B, the hardening rule under such a condition is equivalent to that under duplex slip with an incremental loading $\overline{A'B}$. Under $\overline{A'B}$ the amount of active hardening in system 1 is $A'C$, and that in system 2 is $A'D$. These are indeed the amount of total active hardening in both slip systems if $\overline{AB}$ is treated separately as described above. On the other hand if any active slip system becomes inactive at the end of loading its subsequent yield plane will not contain point B. Under this condition this simplification will be inadequate.

4. COMPARISON WITH PRAGER'S KINEMATIC HARDENING RULE

Prager's kinematic hardening rule, independently proposed by Ishlinskii[23], and the isotropic hardening rule are among the most widely used hardening rules in plasticity. Though Prager's hardening rule was originally proposed for the study of metals, it was used, though without physical justifications, by Budiansky and Wu[4] to describe the hardening behavior of single crystals. This hardening rule states that yield surface moves by a rigid body translation. When the yield surface at the loading point is smooth it moves along the direction of its normal. If the loading point is at the corner of the yield surface then it moves along the direction of the incremental loading. The former condition corresponds to single slip while the latter one corresponds to multiple slip.

First consider the hardening rule developed for the condition of single slip. The subsequent yield planes following a single slip are given by eqn (6). The envelope of these planes is the subsequent yield surface. Since $\cos \overset{(k,1)}{\psi} = \overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}\,\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}$, eqn (6) can be rewritten as

$$\overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}[\sigma_{ij} - \Delta\tau\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}] = \tau_0. \tag{17}$$

Since $\Delta\tau\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}$ is a constant value for all slip systems, comparison between eqn (17) and eqn (2a) indicates that the subsequent yield surface is obtained through the initial yield surface by a rigid body translation along $\Delta\tau\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}$. Since $\overset{(1)}{Q}_{2i}\overset{(1)}{Q}_{1j}$ is normal to the initial yield plane of the active slip system, it can be concluded that under single slip the hardening rule developed coincides with Prager's kinematic hardening rule for a "regular" yield surface.

Next consider the case of multiple slip, or loading at the corner of the yield surface. According to the present theory of multiple slip, the subsequent yield surface is given by eqn (15). Since $\cos \overset{(k,n)}{\psi} = \overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}\overset{(n)}{Q}_{2i}\overset{(n)}{Q}_{1j}$, this equation can be rewritten as

$$\overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}\left[\sigma_{ij} - \sum_{n=1}^{n} \Delta\tau_n \overset{(n)}{Q}_{2i}\overset{(n)}{Q}_{1j}\right] = \tau_0. \tag{18}$$

The decomposition of $\Delta\sigma_{ij}$ into several components along the normals of the yield planes implies that $\sum_{n=1}^{N} \Delta\tau_n \overset{(n)}{Q}_{2i}\overset{(n)}{Q}_{1j} = \Delta\sigma_{ij}$. Equation (18) finally becomes

$$\overset{(k)}{Q}_{2i}\overset{(k)}{Q}_{1j}[\sigma_{ij} - \Delta\sigma_{ij}] = \tau_0. \tag{19}$$

Comparison of eqn (19) with eqn (2a) indicates that the subsequent yield surface under multiple slip can be obtained through a rigid body translation of the initial yield surface along $\Delta\sigma_{ij}$. This result is identical to Prager's hardening rule when the yield surface is "singular" at the loading point.

In a series of experimental investigation on the subsequent yield surfaces of metals, Phillips and his associates[24, 25] have consistently found that yield surface moves along the direction of incremental stress, not along its normal. These observations were used to formulate a hardening rule by Phillips and Weng[26]. Since single crystals tend to undergo multiple slip in the polycrystalline aggregate, the theory developed here serves to provide the physical basis for their observations and hardening rule.

5. CONCLUSION AND DISCUSSION

Based on the theories of Seeger's dislocation pile-up and Orowan's dispersion hardening, the amount of latent hardening was found to be $\Delta\tau \cos\theta \cos\phi$ under single slip and $\sum_{n=1}^{N} \Delta\tau_n \cos\overset{(n)}{\theta} \cos\overset{(n)}{\phi}$ under multiple slip. This hardening law was used to study the motion of yield surface in the stress space. The derived hardening rule was found to coincide with Prager's kinematic hardening rule. Under single slip it moves along the direction of its normal; under multislip it moves along the direction of incremental stress.

The theory developed is consistent with the Bauschinger effect; it is also in good agreement with the observations of Edwards and Washburn on the latent hardening of zinc, cadmium and copper crystals. It is believed that, in the study of the elastic-plastic behavior of a polycrystalline aggregate this theory would provide a more adequate hardening law to its constituent single crystals.

It should be pointed out, however, that Seeger's dislocation pile-up and Orowan's dispersion hardening are not the only hardening mechanisms in single crystals. For a more rigorous account the additional effects on the hardening law due to other possible mechanisms should also be included.

An important implication of the present theory is that the flow stress in the latent system (latent hardening) is always lower than that in the active system. From a typical stress-strain curve of changing slip systems (see Fig. 3) it is noted that the immediate flow stress of the latent system is indeed lower than that of the active one but that its flow stress quickly exceeds that of the active system following the plastic deformation. The latter phenomenon sometimes is stated that the amount of latent hardening is higher than the amount of active hardening. Under the present context it is more appropriate to state that such a phenomenon means that the latent hardening rate is higher than the active hardening rate. Because the dislocation movement in the newly active slip system can lead to the formation of additional sessile dislocations [15] it is conceivable that the latent hardening rate can be higher than the active one and that plastic flow can lead to a higher flow stress in the latent system at a later stage. But the presence of back stress definitely implies that the immediate flow stress in the active system is higher than that in the latent one.

Acknowledgement—This work was supported by the United States National Science Foundation, Solid Mechanics Program, under Grant ENG-78-09738.

REFERENCES

1. G. I. Taylor, Plastic strain in metals. *J. Inst. Metals* **62**, 307 (1938).
2. S. B. Batdorf and B. Budiansky, A mathematical theory of plasticity based on the concept of slip. *NACA TN*-1871 (1949).
3. E. Kröner, Zur Plastischen Verformung des Vielkristalls. *Acta Met.* **9**, 155 (1961).
4. B. Budiansky and T. T. Wu, Theoretical prediction of plastic strains of polycrystals. *Proc. 4th U.S. Natl. Cong. Appl. Mech.* 1175 (1962).
5. R. Hill, Continuum micro-mechanics of elastoplastic polycrystals. *J. Mech. Phys. Solids* **13**, 89 (1965).
6. J. W. Hutchinson, Elastic-plastic behavior of polycrystalline metals and composites. *Proc. R. Soc. Lond.* **A319**, 247 (1970).
7. T. H. Lin, Physical theory of plasticity, *Adv. Appl. Mech.* **11**, 256 (1971).
8. W. T. Koiter, Stress-strain relations, uniqueness and variational theorems for elastic-plastic materials with a singular yield surface. *Quart. Appl. Math.* **11**, 350 (1953).
9. R. Hill, Generalized constitutive relations for incremental deformation of metal crystals by multislip. *J. Mech. Phys. Solids* **14**, 95 (1966).
10. E. H. Edwards, J. Washburn and E. R. Parker, Some observations on the work hardening of metals. *Trans. AIME* **197**, 1525 (1953).
11. N. J. Wadsworth, Work hardening of copper crystals under cyclic straining. *Acta Met.* **11**, 663 (1963).
12. E. Orowan, Discussions. In *Symposium on Internal Stress in Metals and Alloys, Inst. Metals*, London, p. 451 (1947).
13. E. Orowan, Causes and effects of internal stresses. In *Internal Stresses and Fatigue in Metals* (Edited by G. Rassweiler and W. Grube), p. 59. Elsevier, Amsterdam (1959).
14. G. J. Weng and A. Phillips, An investigation of yield surfaces based on dislocation mechanics—I. Basic theory. *Int. J. Engng Sci.* **15**, 45 (1977).
15. A. Seeger, Glide and work hardening in face-centered cubic and hexagonal close-packed metals. In *Dislocations and Mechanical Properties of Crystals* (Edited by J. C. Fisher *et al.*), p. 243. Wiley, New York (1957).

16. W. Prager, The theory of plasticity: a survey of recent achievements (James Clayton Lecture). *Proc. Inst. Mech. Engrs* **169**, 41 (1955).
17. A. Seeger, J. Diehl, S. Mader and H. Rebstock, Work-hardening and work-softening of face-centered cubic metal crystals. *Phil. Mag.* **8-2-15**, 323 (1957).
18. G. I. Taylor, The mechanism of plastic deformation of crystals. *Proc. R. Soc. Lond.* **A165**, 362 (1934).
19. G. E. Dieter, Jr., *Mechanical Metallurgy*, p. 112. McGraw-Hill, New York (1961).
20. M. Peach and J. S. Koehler, The forces exerted on dislocations and the stress fields produced by them. *Phys. Rev.* **80-3**, 436 (1950).
21. E. H. Edwards and J. Washburn, Strain hardening of latent slip systems in zinc crystals. *Trans. AIME* **200**, 1239 (1954).
22. D. C. Drucker, On the postulate of stability of material in the mechanics of continua. *J. de Mecanique* **3-2**, 235 (1964).
23. A. Yu Ishlinskii, General theory of plasticity with linear hardening. *Ukr. Math. Zh.* **6**, 314 (1954).
24. A. Phillips and J. L. Tang, The effect of loading path on the yield surface at elevated temperatures. *Int. J. Solids Structures* **8**, 463 (1972).
25. A. Phillips and M. Ricciuti, Fundamental experiments in plasticity and creep of aluminum: extension of previous results. *Int. J. Solids Structures* **12**, 159 (1976).
26. A. Phillips and G. J. Weng, An analytical study of an experimentally verified hardening law. *J. Appl. Mech.* **42**, 375 (1975).